3. DWT (Discrete Wavelet or Framelet Transform)

August-14-08
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1. DWT using the Haar Filter $[\frac{1}{2}\ \frac{1}{2}]$ and $[\frac{1}{2}\ -\frac{1}{2}]$

$\boxed{52\ \ 8}\ \ 0\ \ 4\ \ 10\ \ 12\ \ 22\ \ 16$

$0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7$

$\sqrt{2}\times$ $\frac{52+8}{2}\ \ \frac{0+4}{2}\ \ \frac{10+12}{2}\ \ \frac{22+16}{2}\ \ \ \ \frac{52-8}{2}\ \ \frac{0-4}{2}\ \ \frac{10-12}{2}\ \ \frac{22-16}{2}$

$0\ \ 1\ \ 2\ \ 3\ \ \ \ 0\ \ 1\ \ 2\ \ 3$

$\sqrt{2}\times$ $\frac{52+8+0+4}{4}\ \ \frac{10+12+22+16}{4}\ \ \frac{52+8-0-4}{4}\ \ \frac{10+12-22-16}{2}$

0

Average
(Lowpass coefficients)

differences
(highpass coefficients)

2. Some notations

1. $u = \{u_k\}_{k=-\infty}^{\infty} : \mathbb{Z} \longrightarrow \mathbb{C}$ is a sequence on $\mathbb{Z}$.
2. Its Fourier series $\hat{u}$ is denoted by

$$\hat{u}(\xi) := \sum_{k=-\infty}^{\infty} u_k e^{-ik\xi}, \qquad \xi \in \mathbb{R}.$$

3. A FIR (finite impulse filter) u (or $\hat{u}$) is a finitely supported sequence on $\mathbb{Z}$.

4. Convolution of u and v:

$$[u*v]_j := \sum_{k=-\infty}^{\infty} u_{j-k} v_k. \quad \text{That is, } \widehat{u*v}(\xi) = \hat{u}(\xi)\hat{v}(\xi).$$

5. $\overline{x}$ denotes the complex conjugate of $x \in \mathbb{C}$.

6. The sequence w given by $\hat{w}(\xi) := \overline{\hat{u}(\xi)}\,\hat{v}(\xi)$ is

$$w_j = \sum_{k=-\infty}^{\infty} \overline{u_k}\, v_{j+k}, \quad j \in \mathbb{Z}.$$

7. For $f \in L_1(\mathbb{R})$, its Fourier transform is defined to be

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(t) e^{-it\xi}\, dt, \quad \xi \in \mathbb{R}.$$

8 Convolution of $f, g \in L_1(\mathbb{R})$:

$$f * g(\xi) := \int_{\mathbb{R}} f(\xi - t)\, g(t)\, dt, \quad \xi \in \mathbb{R}.$$

That is, $\widehat{f*g}(\xi) = \hat{f}(\xi)\,\hat{g}(\xi)$.

3 Perfect Reconstruction Wavelet Filters:

primal: lowpass a, highpass $b^1, \dots, b^L$;
Dual : lowpass $\tilde{a}$, highpass $\tilde{b}^1, \dots, \tilde{b}^L$.

Perfect Reconstruction Condition (in the frequency domain):

$$\begin{bmatrix} \overline{\hat{\tilde{a}}(\xi)} & \overline{\hat{\tilde{b}}^1(\xi)} & \cdots & \overline{\hat{\tilde{b}}^L(\xi)} \\ \overline{\hat{\tilde{a}}(\xi+\pi)} & \overline{\hat{\tilde{b}}^1(\xi+\pi)} & \cdots & \overline{\hat{\tilde{b}}^L(\xi+\pi)} \end{bmatrix} \begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi+\pi) \\ \hat{b}^1(\xi) & \hat{b}^1(\xi+\pi) \\ \vdots & \vdots \\ \hat{b}^L(\xi) & \hat{b}^L(\xi+\pi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

That is,

$$\begin{cases} \overline{\hat{\tilde{a}}(\xi)}\, \hat{a}(\xi) + \sum_{\ell=1}^{L} \overline{\hat{\tilde{b}}^\ell(\xi)}\, \hat{b}^\ell(\xi) = 1 \\ \overline{\hat{\tilde{a}}(\xi)}\, \hat{a}(\xi+\pi) + \sum^{L} \overline{\hat{\tilde{b}}^\ell(\xi)}\, \hat{b}^\ell(\xi+\pi) = 0 \end{cases} \quad \forall \xi \in \mathbb{R}$$

In the time domain, this condition is equivalent to:

$$\begin{cases} \sum_{k=-\infty}^{\infty} \overline{\tilde{a}_{j+k}}\, a_k + \sum_{\ell=1}^{L} \sum_{k=-\infty}^{\infty} \overline{\tilde{b}^{\ell}_{j+k}}\, b^{\ell}_k = \delta_j & \forall j \in \mathbb{Z}. \\ \sum_{k=-\infty}^{\infty} \overline{\tilde{a}_{j+k}}\, (-1)^k a_k + \sum_{\ell=1}^{L} \sum_{k=-\infty}^{\infty} \overline{\tilde{b}^{\ell}_{j+k}}\, (-1)^k b^{\ell}_k = 0, & \end{cases}$$

where δ denotes the Dirac sequence such that

$$\delta_0 = 1 \quad \text{and} \quad \delta_j = 0 \quad \forall j \neq 0.$$

This is also equivalent to

$$\begin{cases} \sum_{k=-\infty}^{\infty} \overline{\tilde{a}_{j+2k}}\, a_{2k} + \sum_{\ell=1} \sum_{k=-\infty} \overline{\tilde{b}^{\ell}_{j+2k}}\, b^{\ell}_{2k} = \frac{1}{2}\delta_j & \forall j \in \mathbb{Z}. \\ \sum_{k=-\infty}^{\infty} \overline{\tilde{a}_{j+2k+1}}\, a_{2k+1} + \sum_{\ell=1}^{L} \sum_{k=-\infty}^{\infty} \overline{\tilde{b}^{\ell}_{j+2k+1}}\, b^{\ell}_{2k+1} = \frac{1}{2}\delta_j & \end{cases}$$

4 The special case $L=1$: one lowpass and one highpass

The perfect reconstruction condition becomes

$$\begin{bmatrix} \overline{\hat{\tilde{a}}(\xi)} & \overline{\hat{\tilde{b}}(\xi)} \\ \overline{\hat{\tilde{a}}(\xi+\pi)} & \overline{\hat{\tilde{b}}(\xi+\pi)} \end{bmatrix} \begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi+\pi) \\ \hat{b}(\xi) & \hat{b}(\xi+\pi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is equivalent to (noting that they are square matrices)

$$\begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi+\pi) \\ \hat{b}(\xi) & \hat{b}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \overline{\hat{\tilde{a}}(\xi)} & \overline{\hat{\tilde{b}}(\xi)} \\ \overline{\hat{\tilde{a}}(\xi+\pi)} & \overline{\hat{\tilde{b}}(\xi+\pi)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

That is

$$\text{I.}\quad \boxed{\overline{\hat{a}(\xi)}\;\hat{\tilde{a}}(\xi) + \overline{\hat{a}(\xi+\pi)}\;\hat{\tilde{a}}(\xi+\pi) = 1} \quad (*)$$

$$\text{II.}\quad \begin{cases} \overline{\hat{a}(\xi)}\;\hat{\tilde{b}}(\xi) + \overline{\hat{a}(\xi+\pi)}\;\hat{\tilde{b}}(\xi+\pi) = 0 \\ \overline{\hat{\tilde{a}}(\xi)}\;\hat{b}(\xi) + \overline{\hat{\tilde{a}}(\xi+\pi)}\;\hat{b}(\xi+\pi) = 0 \end{cases}$$

$$\text{III.}\quad \overline{\hat{b}(\xi)}\;\hat{\tilde{b}}(\xi) + \overline{\hat{b}(\xi+\pi)}\;\hat{\tilde{b}}(\xi+\pi) = 0$$

In the time domain, this is equivalent to

Fundamental $\boxed{\sum_{k=-\infty}^{\infty} \overline{a_{2j+k}}\;\tilde{a}_k = \frac{1}{2}\delta_j \quad (a \text{ and } \tilde{a} \text{ are biorthogonal})} \; (*)$

$$\sum_{k=-\infty}^{\infty} \overline{a_{2j+k}}\;\tilde{b}_k = 0 \quad (a \text{ and } \tilde{b} \text{ are orthogonal})$$

$$\sum_{k=-\infty}^{-\infty} \overline{\tilde{a}_{2j+k}}\;b_k = 0 \quad (\tilde{a} \text{ and } b \text{ are orthogonal})$$

$$\sum_{k=-\infty}^{\infty} \overline{b_{2j+k}}\;\tilde{b}_k = \frac{1}{2}\delta_j \quad (b \text{ and } \tilde{b} \text{ are biorthogonal})$$

If the biorthogonal condition in (*) (that is, a and $\tilde{a}$ are biorthogonal), a standard choice to design the highpass filters b and $\tilde{b}$ is

$$\boxed{\begin{aligned} \hat{b}(\xi) &:= e^{-i\xi}\,\overline{\hat{\tilde{a}}(\xi+\pi)}, \qquad b_k = (-1)^k\,\overline{\tilde{a}_{1-k}} \\ \hat{\tilde{b}}(\xi) &:= e^{-i\xi}\,\overline{\hat{a}(\xi+\pi)}. \qquad \tilde{b}_k = (-1)^k\,\overline{a_{1-k}} \end{aligned}}$$

Since it is easy to check that

$$\begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi+\pi) \\ \overline{\hat{b}(\xi)} & \overline{\hat{b}(\xi+\pi)} \end{bmatrix} \begin{bmatrix} \hat{\tilde{a}}(\xi) & \hat{\tilde{b}}(\xi) \\ \hat{\tilde{a}}(\xi+\pi) & \hat{\tilde{b}}(\xi+\pi) \end{bmatrix}$$

$$= \begin{bmatrix} \overline{\hat{a}(\xi)} & \overline{\hat{a}(\xi+\pi)} \\ e^{i\xi}\hat{\tilde{a}}(\xi+\pi) & -e^{i\xi}\hat{\tilde{a}}(\xi) \end{bmatrix} \begin{bmatrix} \hat{\tilde{a}}(\xi) & e^{-i\xi}\overline{\hat{a}(\xi+\pi)} \\ \hat{\tilde{a}}(\xi+\pi) & -e^{i\xi}\overline{\hat{a}(\xi)} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Definition: a is biorthogonal (or dual) to $\tilde{a}$ (or $\tilde{a}$ is a dual mask/filter to a) if

$$\overline{\hat{a}(\xi)}\,\hat{\tilde{a}}(\xi) + \overline{\hat{a}(\xi+\pi)}\,\hat{\tilde{a}}(\xi+\pi) = 1.$$

In particular, if a is a dual mask of itself, then we call a is an orthogonal mask/filter. That is,

$$|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2 = 1.$$

Example. The Haar filter a: $\frac{1}{2}$ at 0, $\frac{1}{2}$ at 1. So

$$\hat{a}(\xi) = \frac{1+e^{-i\xi}}{2} \Rightarrow |\hat{a}(\xi)|^2 = \cos^2(\xi/2).$$

$$\Rightarrow |\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2 = \cos^2(\xi/2) + \sin^2(\xi/2) = 1.$$

Conclusion: The Haar filter is orthogonal. The highpass filter

$$\hat{b}(\xi) = e^{-i\xi}\,\overline{\hat{a}(\xi+\pi)} = \tfrac{1}{2}(e^{-i\xi} - 1)$$

so b: $-\frac{1}{2}$ at 0, $\frac{1}{2}$ at 1

5 The transition operator and the subdivision operator

Let u and v be two sequences on $\mathbb{Z}$.

The transition operator $T_{u,2}$ is defined to be

$$[T_{u,2}v]_j := \sum_{k=-\infty}^{\infty} v_k \overline{u_{k-2j}}, \quad j \in \mathbb{Z}.$$

The subdivision operator $S_{u,2}$ is defined to be

$$[S_{u,2}v]_j := 2\sum_{k=-\infty}^{\infty} v_k u_{j-2k}, \quad j \in \mathbb{Z}.$$

In the frequency domain, we have

$$\widehat{T_{u,2}v}(2\xi) = \frac{1}{2}\left[\hat{v}(\xi)\overline{\hat{u}(\xi)} + \hat{v}(\xi+\pi)\overline{\hat{u}(\xi+\pi)}\right]$$

$$\widehat{S_{u,2}v}(\xi) = 2\hat{v}(2\xi)\hat{u}(\xi).$$

6 The Discrete Wavelet/Framelet Transform:

The Decomposition Transform (or Algorithm):

Input data: $\boxed{c^0}$

$\boxed{c^1 = \sqrt{2}\,T_{\tilde{a},2}c^0}$ Lowpass wavelet coefficients

$\boxed{d^1_1 = \sqrt{2}\,T_{\tilde{b}^1,2}c^0, \ldots, d^1_L = \sqrt{2}\,T_{\tilde{b}^L,2}c^0}$ Highpass wavelet coefficients

The Reconstruction Transform (or Algorithm)

$$\mathring{c}^0 := \frac{1}{\sqrt{2}} S_{a,2}\, c^1 + \frac{1}{\sqrt{2}} \sum_{\ell=1}^{L} S_{b^\ell,2}\, d_\ell^1$$

Theorem (Perfect Reconstruction): $\mathring{c}^0 = c^0$. That is,

$$c^0 = S_{a,2} T_{\bar{a},2} c^0 + \sum_{\ell=1}^{L} S_{b^\ell,2} T_{\bar{b}^\ell,2} c^0 \quad \forall \text{ sequences } c^0$$

if and only if

$$\begin{bmatrix} \overline{\hat{a}(\xi)} & \overline{\hat{b}^1(\xi)} & \cdots & \overline{\hat{b}^L(\xi)} \\ \overline{\hat{a}(\xi+\pi)} & \overline{\hat{b}^1(\xi+\pi)} & \cdots & \overline{\hat{b}^L(\xi+\pi)} \end{bmatrix} \begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi+\pi) \\ \hat{b}^1(\xi) & \hat{b}^1(\xi+\pi) \\ \vdots & \vdots \\ \hat{b}^L(\xi) & \hat{b}^L(\xi+\pi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

proof. Note that

$$\widehat{S_{a,2} T_{\bar{a},2} c^0}(\xi) = 2\, \widehat{T_{\bar{a},2} c^0}(2\xi)\, \hat{a}(\xi)$$
$$= \hat{c}^0(\xi)\overline{\hat{a}(\xi)}\,\hat{a}(\xi) + \hat{c}^0(\xi+\pi)\overline{\hat{a}(\xi+\pi)}\,\hat{a}(\xi)$$

and

$$\widehat{S_{b^\ell,2} T_{\bar{b}^\ell,2} c^0}(\xi) = \hat{c}^0(\xi)\overline{\hat{b}^\ell(\xi)}\,\hat{b}^\ell(\xi) + \hat{c}^0(\xi+\pi)\overline{\hat{b}^\ell(\xi+\pi)}\,\hat{b}^\ell(\xi)$$

Consequently,

$$\widehat{S_{a,2} T_{\bar{a},2} c^0}(\xi) + \sum_{\ell=1}^{L} \widehat{S_{b^\ell,2} T_{\bar{b}^\ell,2} c^0}(\xi)$$
$$= \hat{c}^0(\xi)\Big[\overline{\hat{a}(\xi)}\,\hat{a}(\xi) + \sum_{\ell=1}^{L} \overline{\hat{b}^\ell(\xi)}\,\hat{b}^\ell(\xi)\Big]$$
$$+ \hat{c}^0(\xi+\pi)\Big[\overline{\hat{a}(\xi+\pi)}\,\hat{a}(\xi) + \sum_{\ell=1}^{L} \overline{\hat{b}^\ell(\xi+\pi)}\,\hat{b}^\ell(\xi)\Big]$$

Therefore, the perfect reconstruct property is equivalent to

$$\overline{\hat{a}(\xi)}\,\hat{a}(\xi) + \sum_{\ell=1}^{L} \overline{\hat{b}^\ell(\xi)}\,\hat{b}^\ell(\xi) = 1$$

and

$$\overline{\hat{a}(\xi+\pi)}\,\hat{a}(\xi) + \sum_{\ell=1}^{L} \overline{\hat{b}^\ell(\xi+\pi)}\,\hat{b}^\ell(\xi) = 0.$$

This completes the proof. $\blacksquare$

7 Multilevel Discrete Wavelet/Framelet Transform.

We can perform DWT on the lowpass wavelet coefficients again and again:

$$C^n := \sqrt{2}\; T_{\tilde{a},2}\, C^{n-1}$$

and

$$d_1^n := \sqrt{2}\; T_{\tilde{b}^1,2}\, C^{n-1}, \;\dots,\; d_L^n := \sqrt{2}\; T_{\tilde{b}^L,2}\, C^{n-1}.$$

That is, we have the following diagram:

$$\begin{tikzcd}
C^0 \arrow[r, "L"] \arrow[dr, "H"] & C^1 \arrow[r, "L"] \arrow[dr, "H"] & C^2 \arrow[r, "L"] \arrow[dr, "H"] & C^3 \cdots \\
 & d_1^1 \arrow[d, phantom, "\vdots"] & d_1^2 \arrow[d, phantom, "\vdots"] & d_1^3 \arrow[d, phantom, "\vdots"] \\
 & d_L^1 & d_L^2 & d_L^3
\end{tikzcd}$$

8 Implementation of $T_{u,2}$ and $S_{u,2}$

$*$ $$[T_{u,2}\, v]_j = \sum_{k=-\infty}^{\infty} v_k\, \overline{u_{k-2j}} = \sum_{k=-\infty}^{\infty} v_k\, \overline{u}_{-(2j-k)} = \sum_{k=-\infty}^{\infty} v_k\, u^*_{2j-k}$$

$$= (v * u^*)_{2j}, \quad \text{where} \quad u^*_k = \overline{u_{-k}}$$

That is, $\widehat{u^*}(\xi) = \overline{\hat{u}(\xi)}$. Hence

$$v \xrightarrow{T_{u,2}} T_{u,2}\, v \;\equiv\; v \xrightarrow{*u^*} v * u^* \xrightarrow[\text{by } 2]{\text{down sampling}} (v * u^*)\downarrow_2,$$

where down sampling by 2 is $(v\downarrow_2)_j = v_{2j}$

Note that the support of $(v\downarrow_2)$ is just half of v.

$*$ $$[S_{u,2}\, v]_j = 2 \sum_{k=-\infty}^{\infty} v_k\, u_{j-2k} = 2 \sum_{k=-\infty}^{\infty} (v\uparrow_2)_{2k}\, u_{j-2k}$$

$$= 2\, \big((v\uparrow_2) * u \big)_j,$$

where the up sampling by 2 is

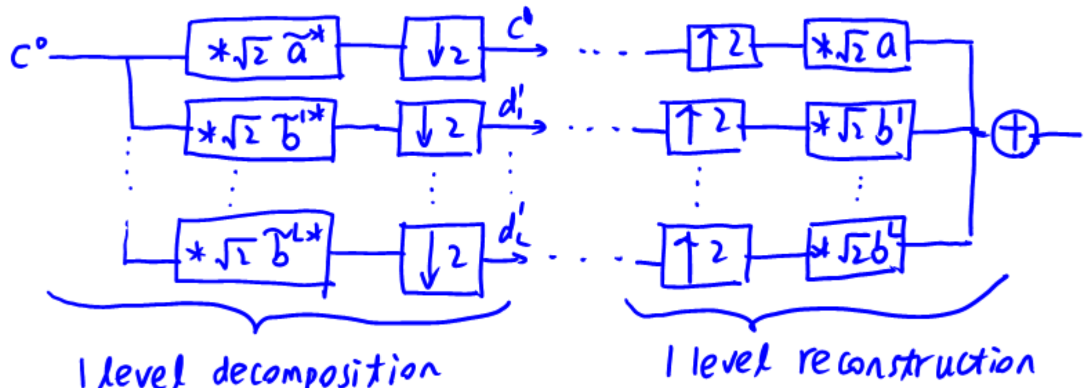
$$(v\uparrow_2)_k = \begin{cases} v_{k/2}, & \text{if } k \text{ is even} \\ 0, & \text{if } k \text{ is odd} \end{cases}$$

ex. $$\begin{array}{ccccc} 9 & -4 & 5 & -3 & 8 \\ \hline -2 & -1 & 0 & 1 & 2 \end{array} \;\xrightarrow[\text{by } 2]{\text{up sampling}}\; \begin{array}{ccccccccc} 9 & 0 & -4 & 0 & 5 & 0 & -3 & 0 & 8 \\ \hline -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \end{array}$$

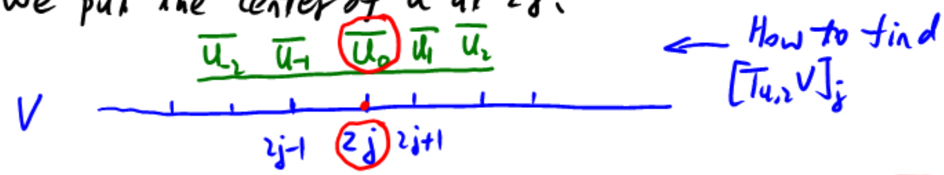
Therefore

$$v \xrightarrow{S_{u,2}} S_{u,2}v \;\equiv\; v \xrightarrow[\text{by } 2]{\text{up sampling}} v\uparrow_2 \xrightarrow{*u} (v\uparrow_2)*u \xrightarrow{\times 2} 2(v\uparrow_2)*u$$

So the Discrete Wavelet transform can be also viewed as



* More intuitively, to calculate $[T_{u,2}v]_j = \sum_{k=-\infty}^{\infty} v_k \overline{u_{k-2j}}$, we put the center of u at $2j$:



Note: No flipping of the filter u, but chang u into $\overline{u}$!

Caution: There are several normalization conditions on the lowpass filter a and $\tilde{a}$. Here we are using $\hat{a}(0) = \hat{\tilde{a}}(0) = 1$. That is, $\sum_{k=-\infty}^{\infty} a_k = \sum_{k=-\infty}^{\infty} \tilde{a}_k = 1$!

DWT for Data on Bounded Interval

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1. Suppose that data v has support $[M_{min}\ \ M_{max}]$ and filter u has support $[N_{min}, N_{max}]$.

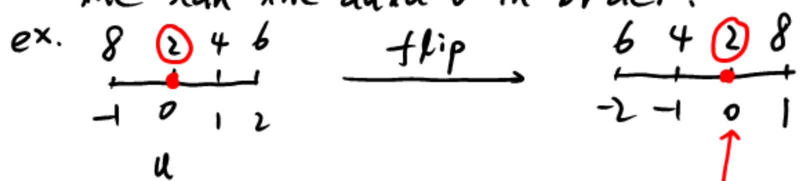
Then $v*u$ has support $[M_{min}-N_{max}, M_{max}-N_{min}]$ and its total length is

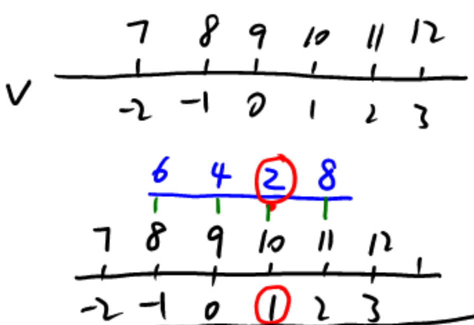
$$(M_{max}-N_{min})-(M_{min}-N_{max}) = (M_{max}-M_{min}) + (N_{max}-N_{min})$$

$$= \text{Length}(v) + \underbrace{\text{Length}(u)}_{\text{This will cause some problem}}$$

Recall that to calculate $v*u$, one first flip the filter u, the scan the data v in order:

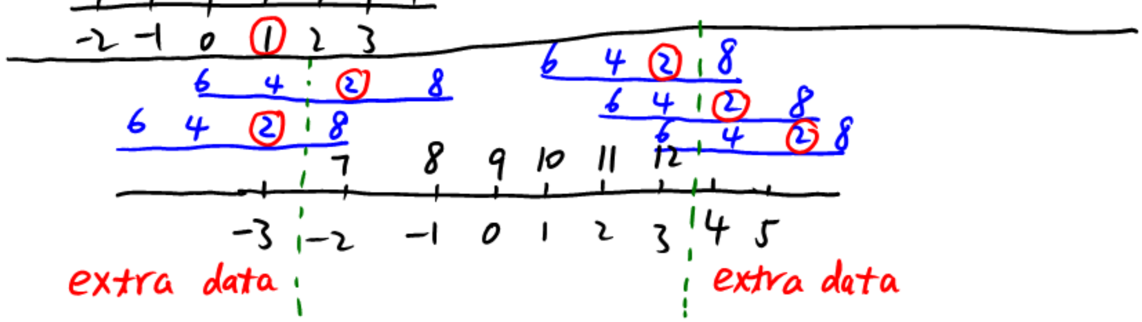
ex.



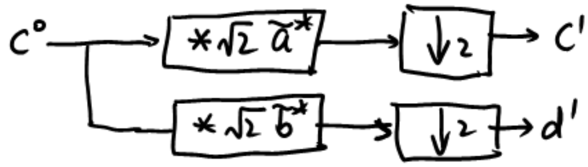


To calculate $(v*u)_1$, put the center of the flipped u at position 1

Then $(v*u)_1 = 6\times8+4\times9+2\times10+8\times11$



In the nonredandant case : $L=1$, we have



Due to down sampling by 2, the $\text{length}(c^1) \approx \frac{1}{2}\text{length}(c^0)$
and the $\text{Length}(d^1) \approx \frac{1}{2}\text{Length}(c^0)$

so $\text{Length}(c^1) + \text{Length}(d^1) \approx \text{Length}(c^0)$.

However, due to the filter length of $\tilde{a}$ and $\tilde{b}$, the exact length relation is

$$\text{Length}(c^1) + \text{Length}(d^1) = \text{Length}(c^0) + \text{LengthExtra},$$

with

LengthExtra more or less only depends on the length of filters $\tilde{a}$ and $\tilde{b}$. Quite often LengthExtra > 0 (Except the Haar filter for $\text{Length}(c^0)$ a power of 2).

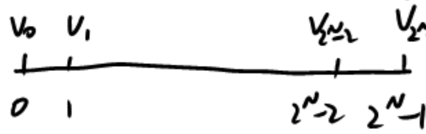
In other words, even for the smallest nonredundant case $L=1$, for data of length n, the total number of all wavelet coefficients $> n$. This is not desirable in many applications: We need extra storage for the extra wavelet coefficients. if we just pad the outside boundary by 0.

How can we handle the extra wavelet coefficients for DWT for data sitting on a bounded interval?

2. Deal data boundary by periodizing the data

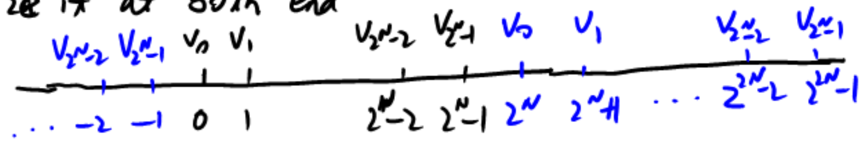
   1. Assume that our data v has support $[0, 2^N-1]$.

   v:



   2. Periodize it at both end

   v^{per}:



   3. Then v^{per} is 2^N-periodic: $v^{per}_{k+2^N} = v^{per}_k$.

   Instead of considering $\sqrt{2}\, T_{\tilde{a},2} v$, we consider

   $$\sqrt{2}\, T_{\tilde{a},2} v^{per}$$

   It is not difficult to check that $\sqrt{2}\, T_{\tilde{a},2} v^{per}$ is 2^{N-1}-periodic.

   4. We use $\sqrt{2}\, T_{\tilde{a},2} v^{per}$ restricted to $[0, 2^{N-1}-1]$ instead of $\sqrt{2}\, T_{\tilde{a},2} v$ which has length $\geq 2^{N-1}$.

   5. If we denote $c^{1,per} := \sqrt{2}\, T_{\tilde{a},2} c^{0,per}\big|_{[0,2^{N-1}-1]}$ and

   $$d^{1,per} := \sqrt{2}\, T_{\tilde{b},2} c^{0,per}\big|_{[0,2^{N-1}-1]}.$$

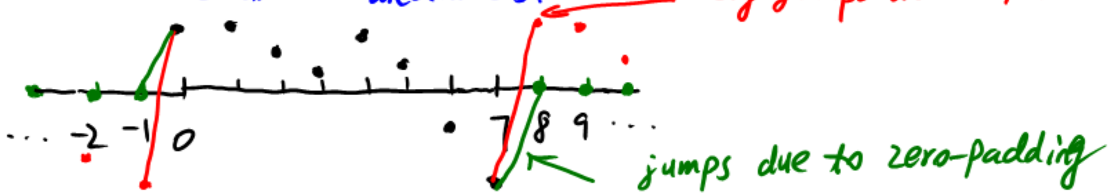
   Then we still keep:

   $$\text{Length}(c^{1,per}) + \text{Length}(d^{1,per}) = 2^N = \text{Length}(c^0)$$

   In essence, we are applying periodic wavelet transform to periodic data!

3. problems about zero-padding and periodization:

We introduce extra jumps/discontinuity at data boundaries.



Big jumps near boundaries introduce large highpass wavelet coefficients near boundaries!

4 linear-phase and symmetry of data and filters.

Let $f \in L_1(\mathbb{R})$ be an integrable function. Its Fourier transform is defined to be:

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(t) e^{-it\xi} d\xi, \qquad \xi \in \mathbb{R}.$$

For any complex number $c \in \mathbb{C}$, we can write

$$c = |c| e^{iP_c}, \quad |c| \geq 0 \text{ is its magnitude and } P_c \in \mathbb{R} \text{ is its phase}$$

For a function $f \in L_1(\mathbb{R})$, we say that f is of (strict) linear phase if

$$\hat{f}(\xi) = |\hat{f}(\xi)| e^{iP_{\hat{f}(\xi)}} \text{ and } P_{\hat{f}(\xi)} \text{ is a linear}$$

function (taking real value), that is,

$$P_{\hat{f}(\xi)} = c_1 + c_2\xi, \quad \forall \xi \in \mathbb{R} \text{ for some } c_1, c_2 \in \mathbb{R}.$$

More generally, we say that f is of linear phase if

$$\hat{f}(\xi) = A(\xi) e^{i(c_1+c_2\xi)} \quad \forall \xi \in \mathbb{R} \text{ for } A(\xi), c_1, c_2 \in \mathbb{R}. \; *$$

That is, we allow the magnitude to be all real numbers.

If (*) holds, then

$$\overline{\hat{f}(\xi)} = A(\xi) e^{-i(c_1+c_2\xi)}.$$

That is, we must have

$$\hat{f}(\xi)\, e^{-i(c_1+c_2\xi)} = A(\xi) = \overline{\hat{f}(\xi)}\, e^{i(c_1+c_2\xi)}.$$

So we must have

$$\overline{\hat{f}(\xi)} = \hat{f}(\xi) e^{-i2(c_1+c_2\xi)}$$

In the time domain, since

$$\widehat{\overline{f(-\cdot)}}(\xi) = \int_{\mathbb{R}} \overline{f(-t)}\, e^{-it\xi} dt = \overline{\int_{\mathbb{R}} f(-t) e^{it\xi} dt}$$

$$= \overline{\int_{\mathbb{R}} f(t) e^{-it\xi} dt} = \overline{\hat{f}(\xi)}$$

$$\widehat{f(\cdot - 2c_2)}(\xi) = \int_{\mathbb{R}} f(t-2c_2) e^{-it\xi} dt = \int_{\mathbb{R}} f(t) e^{-i(t+2c_2)\xi} d\xi$$

$$= e^{-i2c_2\xi}\, \hat{f}(\xi)$$

We have $\widehat{\overline{f(-\cdot)}}(\xi) = e^{-i2c_1}\, \widehat{f(\cdot - 2c_2)}(\xi)$

That is, we must have

$$\overline{f(-t)} = e^{-i2c_1} f(t-2c_2) \quad \forall t \quad (\text{ or a.e. } t)$$

Rewrite it, we have

$$\overline{e^{-ic_1} f(-t)} = e^{-ic_1} f(t-2c_2)$$

Denote $g(t) := e^{-ic_1} f(t)$. Then

$$\overline{g(-t)} = g(t-2c_2) \quad \text{with } g(t) := e^{-ic_1} f(t).$$

Note that f is a real valued function

$$\iff \quad \overline{\hat{f}(\xi)} = \hat{f}(-\xi)$$

Now we discuss linear-phase property for real-valued functions:

By $\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-it\xi} dt$, we have

$$\hat{f}^{(n)}(0) = \int_{\mathbb{R}} f(t)(-it)^n dt = (-i)^n \int_{\mathbb{R}} f(t) t^n dt$$

Assuming $\hat{f}(\xi) = \frac{\hat{f}^{(n)}(0)}{n!} \xi^n + o(|\xi|^n)$, $\xi \to 0$

with $\hat{f}^{(n)}(0) \neq 0$.

By $\quad \overline{\hat{f}(\xi)} = \hat{f}(\xi) e^{-i2(c_1+c_2\xi)}$

we have

$$\overline{\frac{\hat{f}^{(n)}(0)}{n!}}\,\xi^n = \frac{\hat{f}^{(n)}(0)}{n!}\,\xi^n e^{-i(2c_1+c_2\xi)} + o(\xi^n)$$

That is, $\quad \overline{\hat{f}^{(n)}(0)} = \hat{f}^{(n)}(0)\, e^{-i2c_1}$

Since $\quad \hat{f}^{(n)}(0) = (-i)^n \underbrace{\int_{\mathbb{R}} f(t)\, t^n\, dt}_{\text{real number}}$

we must have $\quad \overline{(-i)^n} = (-i)^n e^{-i2c_1} \Longrightarrow$

$$(-1)^n (-i)^n = (-i)^n e^{-i2c_1} \Rightarrow e^{-i2c_1} = (-1)^n$$

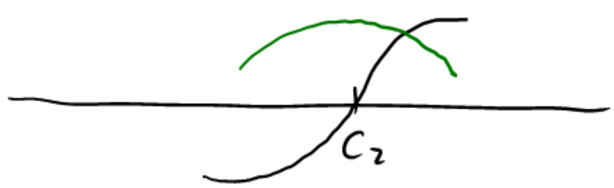
That is $\quad \overline{\hat{f}(\xi)} = \hat{f}(\xi) e^{-i2c_2\xi} \quad$ or $\quad \overline{\hat{f}(\xi)} = -\hat{f}(\xi) e^{-i2c_2\xi}$

In other words,

$$f(-t) = \pm f(t - 2c_2).$$

That is, $\quad f$ is either symmetric or antisymmetric about c_2 by

$$f(-c_2) = \pm f(c_2 - 2c_2) = \pm f(c_2)$$



Conclusion: For a real-value (compactly supported) function f, f is of linear-phase $\Longleftrightarrow$ f has symmetry (symmetric or antisymmetric)

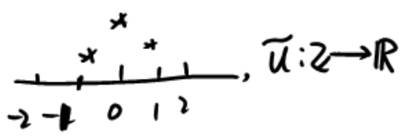
Same conclusion holds for sequences:
For $u : \mathbb{Z} \longrightarrow \mathbb{R}$, u is of linear phase if and only if

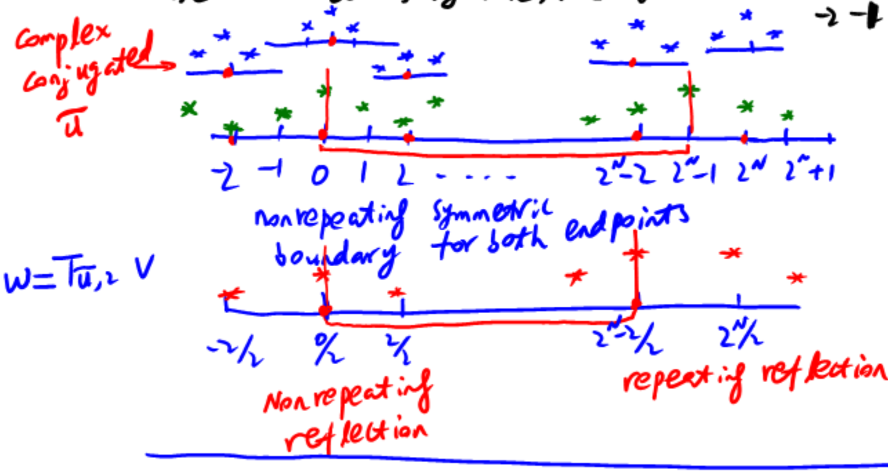
u has one of the following 4 types of symmetry:

linear phase is known to reduce artifact since human eyes are sensitive to non-symmetric objects

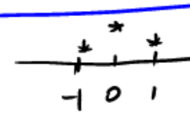
5 DWT for filters with symmetry

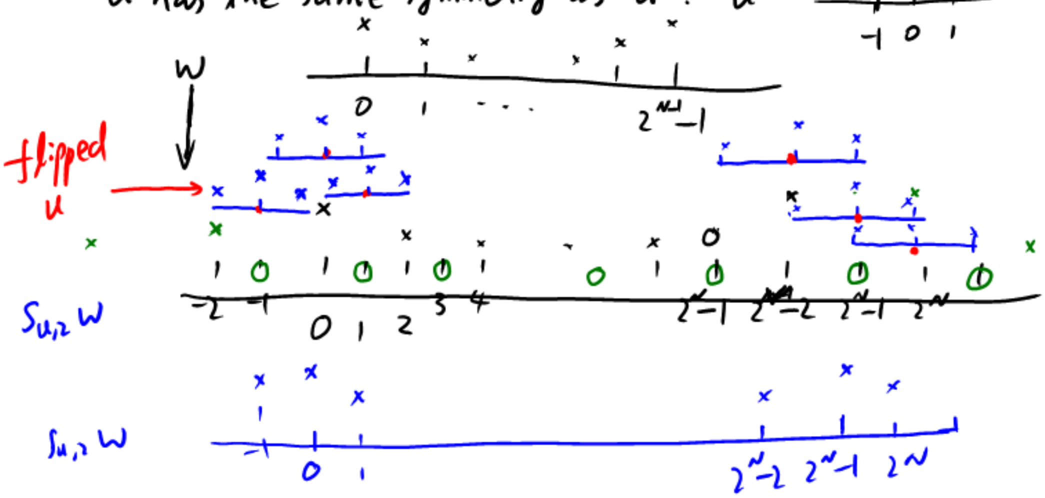
Case 1: $\tilde{u}$ is symmetric about 0



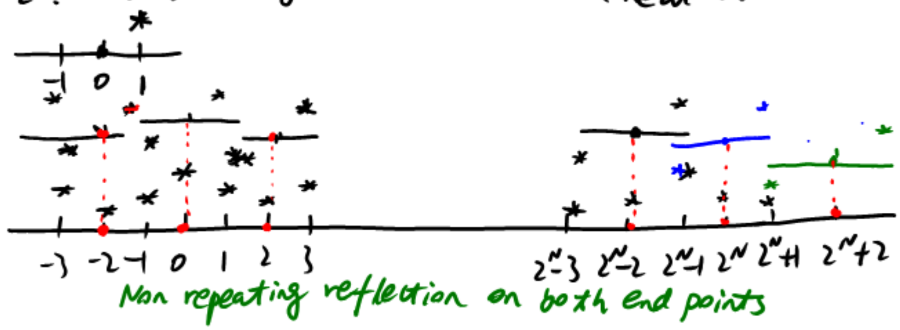


u has the same symmetry as $\tilde{u}$: u

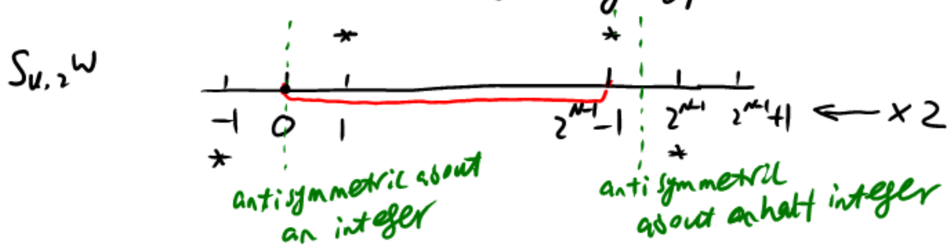




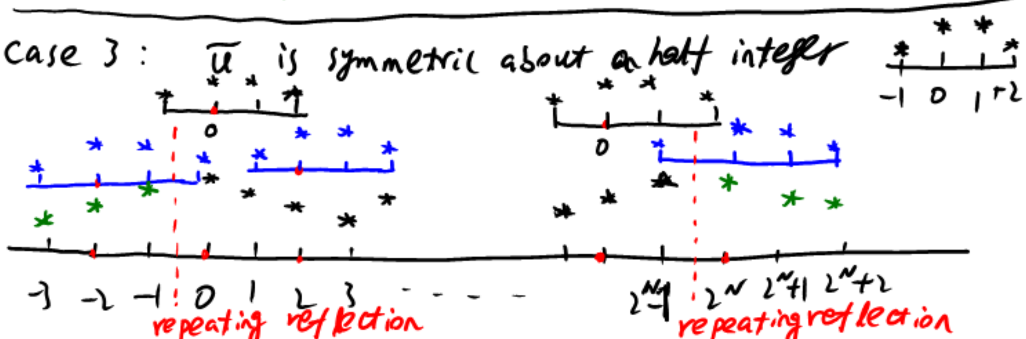
Case 2: u is antisymmetric about 0 (Real-valued filter)



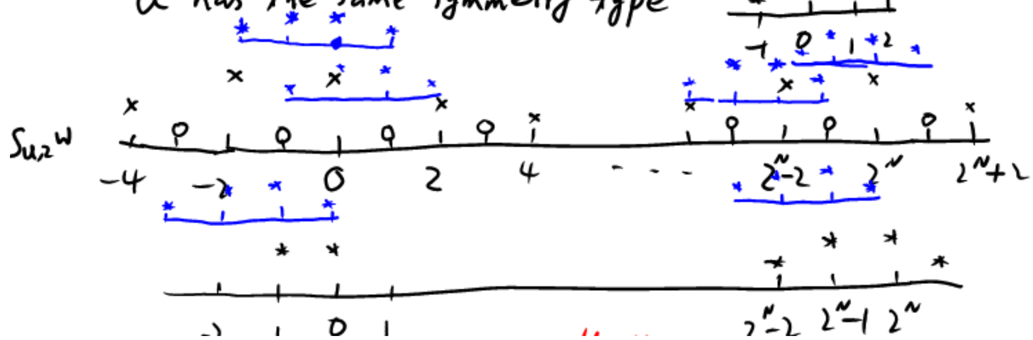
$\tilde{u}$ and u have the same symmetry type



Case 3: $\tilde{u}$ is symmetric about an half integer

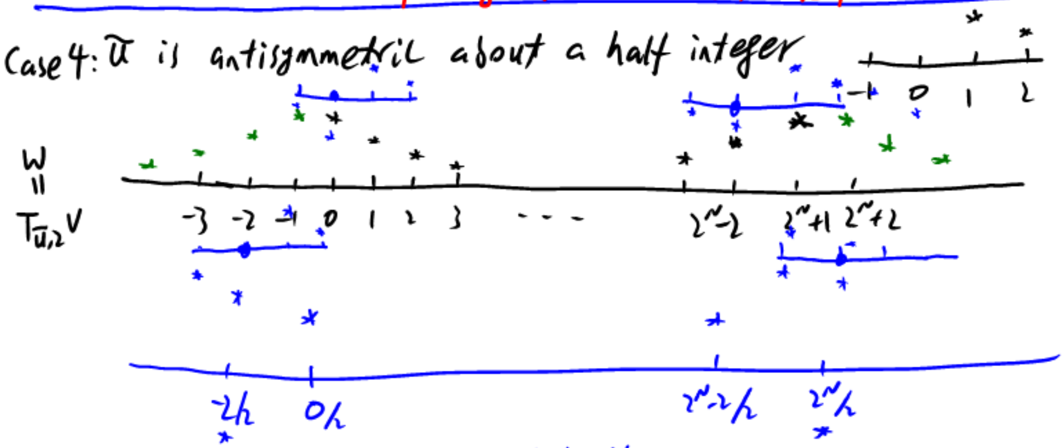


u has the same symmetry type



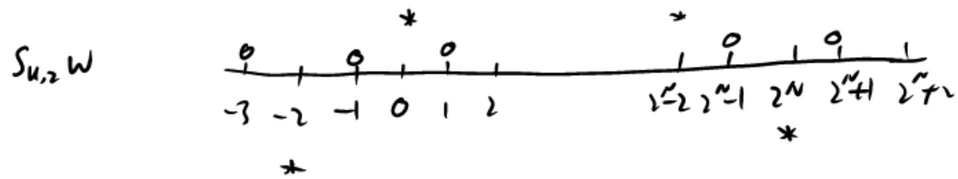
Repeating reflection on both endpoints

Case 4: $\bar{u}$ is antisymmetric about a half integer



antisymmetric at both endpoints

u has the same symmetry type as $\bar{u}$:



Summary